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Analytic Theory of Continued Fractions III

Proceedings of a Seminar-Workshop, held in Redstone, USA, June 26–July 5, 1988

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PREFACE

This volume contains the proceedings of a research Seminar–Workshop held in Redstone, Colorado from June 26 to July 5, 1988. The topic of this workshop was Analytic Theory of Continued Fractions, and it was organized by William B. Jones, University of Colorado and Arne Magnus, Colorado State University.

This was the third workshop of its kind. The first one was held in Loen , Norway in 1981 (proceedings published in LN in M N^0 932), the second one in Pitlochry and Aviemore, Scotland in 1985 (proceedings published in LN in M N^0 1199). The idea is that workers in the field shall come together to exchange ideas and start cooperation. So, in addition to the presentation of our latest results, we have talks on ideas, half–finished projects, ideas that did not work, etc. Questions and comments, stupid or not, are encouraged.

This time analysis of papers from the turn of the century was one of the issues. Mathematicians like Helge von Koch, Jan Sleszynski and Julius Worpitzky had results and arguments which still deserve attention. Indirectly, these studies led to the study of separate convergence, a topic presented in a survey article in this volum, and to a historical article on Julius Worpitzky. In addition we continued discussions on nearness problems, the connection to Padé Approximants, and applications to number theory and differential equations. Interesting in this respect is for instance the use of Lange's δ–fractions to solve Riccati equations, and the question of finding thin subsets E_0 of an element region E whose best limit region V_0 is dense in the best limit region V for E. Also several other topics which are not reflected in these proceedings were discussed and will hopefully be published elsewhere.

It is evident that for workshops like this the location is important. Comfort, "isolation", a good seminar room and soothing nature are essential. We had all this in beautiful Redstone, and we are grateful to Nancy Lambert, manager of the Redstone Inn, for her hospitality. We also wish to express our gratitude to the Norwegian Research Council for Science and the Humanities, to the U.S. National Science Foundation, to the University of Colorado and to the universities of the respective participants for financial support. Finally we would like to thank Professor B. Eckmann for accepting this volume for publication.

Lisa Jacobsen

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δ- Fraction Solutions to Riccati Equations

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1. Introduction

The objective of this paper is to introduce another method for solving scalar Riccati equations by continued fractions. The Riccati differential equation is of particular interest for several reasons: it is one of the simplest nonlinear ordinary differential equations, it is closely associated with a second- order linear differential equation, and it appears in many applications including general relativity [12,13], acoustics [8], systems theory [9], and invariant embedding [1].

Riccati equations have the convenient property that they are invariant (in a sense) under linear fractional transformations (lfts). More specifically, under an lft

$$y = \frac{\alpha(z)w + \beta(z)}{\gamma(z)w + \delta(z)} \tag{1.1}$$

a Riccati equation

$$y' = f_0(z) + f_1(z)y + f_2(z)y^2 \tag{1.2}$$

is transformed into another Riccati equation

$$w' = \tilde{f}_0(z) + \tilde{f}_1(z)w + \tilde{f}_2(z)w^2 \ . \tag{1.3}$$

Since lfts play a fundamental role in the development of continued fractions [7], it is very natural to use continued fractions to solve Riccati equations. There has been a lot of recent interest in the use of continued fractions to solve Riccati equations as evidenced by [2,3,4,5,6,14,15].

Definition 1. *Let D be a (formal) differential operator. A continued fraction with n^{th} approximant $f_n(z)$ is said to be a formal solution of a differential equation $D[W(z)] = 0$ at $z = 0$ if*

$$\Lambda_0\left(D[f_n(z)]\right) = O\left(z^{k_n}\right) \tag{1.4}$$

where $k_n \to \infty$ as $n \to \infty$. Here $\Lambda_0(f)$ denotes the Laurent series about $z = 0$ for a function f meromorphic in a neighborhood of zero.

In this paper a relatively new type of continued fraction, a δ- fraction, is used to find solutions to Riccati differential equations of the form

$$R[W(z)] := A(z) + B(z)W(z) + C(z)W^2(z) - W'(z) = 0 \tag{1.5}$$

and the form

$$R[W(z)] := zA(z) + B(z)W(z) + C(z)W^2(z) - z^k W'(z) = 0 \tag{1.6}$$

under the conditions that $W(0) = 0$, $A(z)$, $B(z)$, and $C(z)$ are analytic at $z = 0$, and in (1.6), $k \in \mathbf{Z}^+$.

The class of δ- fractions was introduced by Lange in 1981 [10]. This is a class of continued fractions whose members are finite or infinite continued fractions of the form

$$b_0 - \delta_0 z + \frac{d_1 z}{1 - \delta_1 z} + \frac{d_2 z}{1 - \delta_2 z} + \cdots \tag{1.7}$$

where b_0 and d_n are complex constants, $d_n \neq 0$ for $n = 1, 2, \ldots$, and the δ_n are real constants equal to either 0 or 1. The δ- fraction is *regular* in case $d_{n+1} = 1$ whenever $\delta_n = 1$. Lange chose the name δ- fraction "because of the binary "impulse" nature of the sequence $\{\delta_n\}$ and the analogies, therefore, with the δ's in the Dirac delta function and the Kronecker delta symbol" [11]. His initial desire was to find a simple class of continued fractions of the form

$$b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \cdots \tag{1.8}$$

that had the following properties.

(i) The elements $a_n(z)$ and $b_n(z)$ are polynomials in z of degree ≤ 1.

(ii) The regular C- fractions

$$d_0 + \frac{d_1 z}{1} + \frac{d_2 z}{1} + \cdots, \quad d_n \in \mathbf{C}, \quad d_n \neq 0 \text{ if } n \geq 1 \tag{1.9}$$

are in the class.

(iii) Given a power series

$$L_0 = c_0 + c_1 z + c_2 z^2 + \cdots, \qquad c_n \in \mathbf{C} \tag{1.10}$$

there exists a unique member of the class that corresponds to L_0.

(iv) If L_0 represents a rational function in a neighborhood of $z = 0$, then its corresponding continued fraction terminates.

(v) For many classical functions analytic in a neighborhood of $z = 0$, the corresponding continued fraction has elements that can be expressed in closed form.

(vi) Convergence results can be obtained.

(vii) In many cases the approximants of the continued fraction corresponding to L_0 appear in the Padé table for L_0.

Many useful continued fractions, among which are C- fractions, general T- fractions, and P-fractions, satisfy some but not all of the requirements (i)- (vii). In [10,11], Lange has shown that the δ- fractions satisfy conditions (i) - (vi) and he has indicated that it is reasonable to expect that they also satisfy condition (vii). Two useful theorems from [11] that will be used in the sequel are the following.

Theorem 2. *Every regular δ- fraction (1.7) corresponds to a unique power series*

$$L_0 = c_0 + c_1 z + c_2 z^2 + \cdots . \tag{1.11}$$

Conversely, for every formal power series (fps) (1.11) there exists a unique δ- fraction which corresponds to it. In the case of the infinite δ- fraction

$$-\delta_0 z + \frac{d_1 z}{1 - \delta_1 z} + \frac{d_2 z}{1 - \delta_2 z} + \cdots \tag{1.12}$$

the order of correspondence for the k^{th} approximant is $k + 1$. In the case of the finite δ-fraction

$$-\delta_0 z + \frac{d_1 z}{1 - \delta_1 z} + \cdots \frac{d_{n-1} z}{1 - \delta_{n-1} z} + \frac{d_n z}{1} \tag{1.13}$$

the order of correspondence for the k^{th} approximant is $k+1$ if $0 \leq k < n$ and ∞ if $k \geq n$.

Theorem 3. *A power series (1.11) is the Taylor series about the origin of a rational function if and only if there exists a finite regular δ- fraction that corresponds to it.*

In section 2, an algorithm is given for constructing δ- fraction solutions to Riccati equations of the forms given in (1.5) and (1.6). In section 3, theoretical results are given. The δ-fraction is shown to be a formal solution to the Riccati equation at $z = 0$ and the connection between the δ- fraction solution and the formal power series solution is given. The last theorem provides the link between the δ- fraction solution and a possible analytic solution. Section 4 is devoted to the computational aspects of the δ- fraction solution. In [3], C- fraction solutions were studied and some comments on the computational aspects were given. C-fraction solutions are compared to δ- fraction solutions and reasons are given for preferring δ- fractions over C- fractions.

2. Algorithm

In this section an algorithm is presented for generating regular δ- fraction solutions to initial value problems involving both nonsingular Riccati equations of the form

$$R[W(z)] := A(z) + B(z)W(z) + C(z)W^2(z) - W'(z) = 0 \tag{2.1}$$

and singular Riccati equations of the form

$$R[W(z)] := zA(z) + B(z)W(z) + C(z)W^2(z) - z^kW'(z) = 0, \tag{2.2}$$

with $W(0) = 0$. The following definition identifies a class of Riccati equations in which every member is guaranteed a regular δ- fraction solution.

Definition 4. *A Riccati equation $R[W(z)] = 0$ is admissible if it satisfies the conditions in either (A.) or (B.).*

(A.) It is of the form (2.1) and $A(z)$, $B(z)$, and $C(z)$ are analytic at $z = 0$.

(B.) It is of the form (2.2) and it satisfies the following requirements.

$$\begin{cases} (i)\, A(z),\ B(z),\ \text{and } C(z) \text{ are analytic at } z = 0. \\ (ii)\, B(0) \text{ and } C(0) \text{ are not both zero.} \\ (iii)\, \text{The constant } k \text{ is a positive integer.} \\ (iv)\, \text{If } k = 1, B(0) \text{ is not a positive integer, and if } k > 1, B(0) \neq 0. \end{cases} \tag{2.3}$$

Theorem 5. *From every admissible Riccati equation it is possible to construct either a finite regular δ- fraction*

$$-\delta_0 z + \frac{d_1 z}{1 - \delta_1 z} + \cdots + \frac{d_{n-1} z}{1 - \delta_{n-1} z} + \frac{d_n z}{1} \tag{2.4}$$

$\delta_k \in \{0, 1\}$ for $k = 0, 1, \ldots, n$ and $d_k \in \mathbf{C}\backslash\{0\}$ for $k = 1, 2, \ldots, n$ or an infinite regular δ- fraction

$$-\delta_0 z + \frac{d_1 z}{1 - \delta_1 z} + \cdots \frac{d_n z}{1 - \delta_n z} + \cdots \tag{2.5}$$

$\delta_k \in \{0,1\}$ for $k = 0,1,\ldots$ and $d_k \in \mathbf{C}\backslash\{0\}$ for $k = 1,2,\ldots$.

Proof: First consider admissible Riccati equations of the form (2.1). A regular δ- fraction will be generated by a process involving the following substitutions,

$$(2.6) \qquad W_0(z) = -\delta_0 z + W_1(z), \qquad W_n(z) = \frac{d_n z}{1 - \delta_n z + W_{n+1}(z)}, \quad n \in \mathbf{Z}^+.$$

A sequence of Riccati equations will also be generated from which the constants δ_k, $k \in \mathbf{Z}_0^+$ and d_k, $k \in \mathbf{Z}^+$, can be determined by forcing the equations to be admissible. Let $W(z) = W_0(z)$, $A(z) = A_0(z)$, $B(z) = B_0(z)$, and $C(z) = C_0(z)$. Starting with $W_0(z) = -\delta_0 z + W_1(z)$ define

$$(2.7) \qquad R_1[W_1(z)] := R_0[-\delta_0 z + W_1(z)] = R_0[W_0(z)]$$

so that

$$(2.8) \qquad R_1[W_1(z)] := A_1(z) + B_1(z)W_1(z) + C_1(z)W_1^2(z) - W_1'(z) = 0$$

with

$$(2.9) \qquad \begin{cases} A_1(z) = \delta_0 + A_0(z) - \delta_0 z B_0(z) + \delta_0^2 z^2 C_0(z) \\ B_1(z) = B_0(z) - 2\delta_0 z C_0(z) \\ C_1(z) = C_0(z). \end{cases}$$

Let $A_1^*(z) = A_0(z)$. Define δ_0 as follows:

$$(2.10) \qquad \delta_0 = \begin{cases} 0 & \text{if } A_1^*(0) \neq 0 \\ 1 & \text{if } A_1^*(0) = 0. \end{cases}$$

Clearly $W_0(z) \equiv 0$ is a solution in case $A_0(z) \equiv 0$, so terminate the process immediately in this case. Otherwise, continue.

The next transformation is $W_1(z) = \dfrac{d_1 z}{1 - \delta_1 z + W_2(z)}$ from which $R_2[W_2(z)]$ is defined

$$(2.11) \qquad R_2[W_2(z)] := \frac{(1 - \delta_1 z + W_2(z))^2}{-d_1} R_1\left[\frac{d_1 z}{1 - \delta_1 z + W_2(z)}\right]$$

$$= \frac{(1 - \delta_1 z + W_2(z))^2}{-d_1} R_1[W_1(z)]$$

so that

$$R_2[W_2(z)] = zA_2(z) + B_2(z)W_2(z) + C_2(z)W_2^2(z) - zW_2'(z) = 0 \tag{2.12}$$

with

$$\begin{cases} zA_2(z) = 1 - \dfrac{1}{d_1}A_1(z) + 2\dfrac{\delta_1}{d_1}zA_1(z) - \dfrac{\delta_1^2}{d_1}z^2A_1(z) - zB_1(z) \\ \qquad +\delta_1 z^2 B_1(z) - d_1 z^2 C_1(z) \\ B_2(z) = 1 - \dfrac{2}{d_1}A_1(z) + 2\dfrac{\delta_1}{d_1}zA_1(z) - zB_1(z) \\ C_2(z) = -\dfrac{1}{d_1}A_1(z). \end{cases} \tag{2.13}$$

Let $zA_2^*(z) = 1 - \frac{1}{d_1}A_1(z) - zB_1(z) - d_1z^2C_1(z)$. In order for $A_2(z)$ to be analytic at $z = 0$, d_1 must be determined by

$$d_1 = A_1(0) \neq 0. \tag{2.14}$$

Note that if $\delta_0 = 1$, then $d_1 = 1$. Determine δ_1 as follows,

$$\delta_1 = \begin{cases} 0 & \text{if } A_2^*(0) \neq 0 \text{ or } A_2^*(z) \equiv 0 \\ 1 & \text{if } A_2^*(0) = 0, \text{but } A_2^*(z) \not\equiv 0. \end{cases} \tag{2.15}$$

The process terminates if $A_2^*(z) \equiv 0$ yielding the finite δ- fraction

$$-\delta_0 + \frac{d_1 z}{1}. \tag{2.16}$$

Otherwise, note that $A_2(0) \neq 0$, $B_2(0) = C_2(0) = -1$.

For $n \geq 2$, use the transformation $W_n(z) = \dfrac{d_n z}{1 - \delta_n z + W_{n+1}}$ to define $R_{n+1}[W_{n+1}]$ as

$$\begin{aligned} R_{n+1}[W_{n+1}] &:= \frac{(1-\delta_n z + W_{n+1}(z))^2}{-d_n z} R_n\left[\frac{d_n z}{1-\delta_n z + W_{n+1}(z)}\right] \\ &= \frac{(1-\delta_n z + W_{n+1}(z))^2}{-d_n z} R_n[W_n(z)]. \end{aligned} \tag{2.17}$$

Thus,

$$R_{n+1}[W_{n+1}] := zA_{n+1}(z) + B_{n+1}(z)W_{n+1}(z) + C_{n+1}(z)W_{n+1}^2(z) - zW_{n+1}'(z) = 0 \tag{2.18}$$

with

$$(2.19)\qquad \begin{cases} zA_{n+1}(z) = 1 - \dfrac{1}{d_n}A_n(z) + 2\dfrac{\delta_n}{d_n}zA_n(z) - \dfrac{\delta_n^2}{d_n}z^2A_n(z) - B_n(z) \\ \qquad +\delta_n zB_n(z) - d_n zC_n(z) \\ B_{n+1}(z) = 1 - \dfrac{2}{d_n}A_n(z) + 2\dfrac{\delta_n}{d_n}zA_n(z) - B_n(z) \\ C_{n+1}(z) = -\dfrac{1}{d_n}A_n(z). \end{cases}$$

Let

$$(2.20)\qquad zA^*_{n+1}(z) = 1 - \frac{1}{d_n}A_n(z) - B_n(z) - d_n zC_n(z).$$

In order for $A_{n+1}(z)$ to be analytic at $z = 0$, we must have

$$(2.21)\qquad d_n = \frac{A_n(0)}{1 - B_n(0)},$$

where it is noted below that $B_n(0) \neq 1$. Determine δ_n as follows,

$$(2.22)\qquad \delta_n = \begin{cases} 0 & \text{if } A^*_{n+1}(0) \neq 0 \text{ or } A^*_{n+1}(z) \equiv 0 \\ 1 & \text{if } A^*_{n+1}(0) = 0, \text{but } A^*_{n+1}(z) \not\equiv 0. \end{cases}$$

The process terminates if $A^*_{n+1}(z) \equiv 0$ yielding the finite δ- fraction

$$(2.23)\qquad -\delta_0 z + \frac{d_1 z}{1 - \delta_1 z} + \cdots + \frac{d_{n-1}z}{1 - \delta_{n-1}z} + \frac{d_n z}{1}.$$

Otherwise the process continues. Note that $A_{n+1}(0) \neq 0$, $B_{n+1}(0) = -C_{n+1}(0) = B_n(0) - 1 = -n$. Notice also that $\delta_n = 1$ implies that $d_{n+1} = 1$ for $n = 1, 2, \ldots$, so the δ- fraction that is generated is a *regular* δ- fraction.

Now consider admissible Riccati equations of the form (2.2). The construction is very similar to that given in the first half, but here there are extra admissibility conditions that must be relied upon. The first substitution is $W_0(z) = -\delta_0 z + W_1(z)$ which is used to define

$$(2.24)\qquad R_1[W_1(z)] := R_0[-\delta_0 z + W_1(z)] = R_0[W_0(z)]$$

so that

$$(2.25)\qquad R_1[W_1(z)] := zA_1(z) + B_1(z)W_1(z) + C_1(z)W_1^2(z) - z^k W_1'(z) = 0$$

where

$$(2.26)\qquad \begin{cases} zA_1(z) = \delta_0 z^k + zA_0(z) - \delta_0 z B_0(z) + \delta_0^2 z^2 C_0(z) \\ B_1(z) = B_0(z) - 2\delta_0 z C_0(z) \\ C_1(z) = C_0(z). \end{cases}$$

Let $zA_1^*(z) = zA_0(z)$. Define δ_0 as follows:

$$(2.27)\qquad \delta_0 = \begin{cases} 0 & \text{if } A_1^*(0) \neq 0 \\ 1 & \text{if } A_1^*(0) = 0. \end{cases}$$

Clearly, $W_0(z) \equiv 0$ is a solution in case $A_0(z) \equiv 0$, so terminate the process immediately in this case. Otherwise, continue. Before proceeding, note that (2.3) guarantees that $A_1(0) \neq 0$.

For $n \geq 2$, use the transformation $W_n(z) = \dfrac{d_n z}{1 - \delta_n z + W_{n+1}(z)}$ to define $R_{n+1}[W_{n+1}]$ as

$$(2.28)\qquad \begin{aligned} R_{n+1}[W_{n+1}] &:= \frac{(1 - \delta_n z + W_{n+1}(z))^2}{-d_n z} R_n \left[\frac{d_n z}{1 - \delta_n z + W_{n+1}(z)} \right] \\ &= \frac{(1 - \delta_n z + W_{n+1}(z))^2}{-d_n z} R_n[W_n(z)]. \end{aligned}$$

A routine calculation produces

$$(2.29)\qquad R_{n+1}[W_{n+1}] = zA_{n+1}(z) + B_{n+1}(z)W_{n+1}(z) + C_{n+1}(z)W_{n+1}^2(z) - z^k W_{n+1}'(z) = 0$$

where

$$(2.30)\qquad \begin{cases} zA_{n+1}(z) = z^{k-1} - \dfrac{1}{d_n} A_n(z) + 2\dfrac{\delta_n}{d_n} z A_n(z) - \dfrac{\delta_n^2}{d_n} z^2 A_n(z) - B_n(z) \\ \qquad\qquad + \delta_n z B_n(z) - d_n z C_n(z) \\ B_{n+1}(z) = z^{k-1} - \dfrac{2}{d_n} A_n(z) + 2\dfrac{\delta_n}{d_n} z A_n(z) - B_n(z) \\ C_{n+1}(z) = -\dfrac{1}{d_n} A_n(z). \end{cases}$$

Let

$$(2.31)\qquad zA_{n+1}^*(z) = z^{k-1} - \frac{1}{d_n} A_n(z) - B_n(z) - d_n z C_n(z).$$

In order for $A_{n+1}(z)$ to be analytic at $z = 0$, we must have

$$(2.32)\qquad d_n = \begin{cases} \dfrac{A_n(0)}{1 - B_n(0)} & \text{if } k = 1 \\ \dfrac{-A_n(0)}{B_n(0)} & \text{if } k > 1. \end{cases}$$

Determine δ_n as follows,

$$(2.33)\qquad \delta_n = \begin{cases} 0 & \text{if } A^*_{n+1}(0) \neq 0 \text{ or } A^*_{n+1}(z) \equiv 0 \\ 1 & \text{if } A^*_{n+1}(0) = 0, \text{but } A^*_{n+1}(z) \not\equiv 0. \end{cases}$$

The process terminates if $A^*_{n+1}(z) \equiv 0$ yielding a finite δ- fraction. Otherwise, the process continues. Also, note that $A_{n+1}(0) \neq 0$ and

$$(2.34)\qquad B_{n+1}(0) = C_{n+1}(0) = \begin{cases} B_n(0) - 1 & \text{if } k = 1 \\ B_0(0) & \text{if } k > 1. \end{cases}$$

It is not hard to see that if $\delta_n = 1$ then $d_{n+1} = 1$ and hence the δ- fraction generated is a regular δ- fraction. ∎

The following is a summary of the algorithm for the nonsingular case. The algorithm for the singular case is analogous with only minor modifications required at the beginning.

Specify the number n of terms of the continued fraction to compute
Enter A_0, B_0 and C_0
If $A_0 \equiv 0$, let $\delta_0 = 0$ and stop
Determine δ_0, A_1, B_1 and C_1 by (2.9) and (2.10)
Determine d_1 by (2.14)
Determine A^*_2
If $A^*_2 \equiv 0$, let $\delta_1 = 0$ and stop
Determine δ_1, A_2, B_2 and C_2 by (2.13) and (2.15)
Do $j = 3, \ldots, n + 1$
  Determine d_{j-1} by (2.21)
  Determine A^*_j by (2.20)
  If $A^*_j \equiv 0$, let $\delta_{j-1} = 0$ and stop
  Determine δ_{j-1}, A_j, B_j and C_j by (2.19) and (2.22)

3. Theory of the δ- fraction Solutions

In section 2 an algorithm was given for constructing a regular δ- fraction from a Riccati equation. The purpose of this section is to pursue the theory of the δ- fraction solution. First we point out that the regular δ- fraction resulting from the algorithm developed in the previous section is a formal continued fraction solution to the Riccati equation to which the algorithm was applied. The objective of the next theorem is to show that any formal δ- fraction solution to an admissible Riccati equation corresponds to the unique fps solution that vanishes at $z = 0$. A corollary to that theorem is that the formal δ- fraction solution guaranteed by Theorems 5 and 6 is the unique formal δ- fraction solution that vanishes at $z = 0$. The penultimate theorem states that an admissible Riccati equation has finite δ-fraction solution if and only if the Riccati equation has a rational solution. Finally, we close the section with a theorem that states the connection between a δ- fraction solution and an analytic solution.

Theorem 6. *Let $R_0[W_0(z)] = 0$ be an admissible Riccati equation. The regular δ- fraction constructed by the algorithm given in the proof of Theorem 5 is a formal continued fraction solution to the Riccati equation.*

Proof. We will prove it for the admissible Riccati equations of the form (2.1). The proof of the other case is completely analogous. From equations (2.7), (2.11), and (2.17) we have

$$(3.1)\qquad \begin{cases} R_0[W_0(z)] = R_1[W_1(z)] = \dfrac{-d_1}{(1-\delta_1 z + W_2(z))^2} R_2[W_2(z)] \\ \qquad = \displaystyle\prod_{k=1}^{n} \left(\frac{-d_k}{(1-\delta_k z + W_{k+1})^2} \right) z^{n-1} R_{n+1}[W_{n+1}(z)]. \end{cases}$$

Also, $W_0(z)$ is related to $W_1(z), W_2(z), \ldots$ by

$$(3.2)\qquad W_0(z) = -\delta_0 z + W_1(z)$$

and

$$(3.3)\qquad W_0(z) = -\delta_0 z + \frac{d_1 z}{1-\delta_1 z} + \cdots + \frac{d_n z}{1-\delta_n z + W_{n+1}(z)}, \quad \text{for } n \geq 2$$

so setting $W_{n+1}(z) = 0$ determines $W_0(z)$. In fact, setting $W_{n+1}(z) = 0$ forces $W_0(z) = f_n$ where f_n is the n^{th} approximant of the δ- fraction solution. Using this result with (3.1) we have

$$(3.4)\qquad R_0[f_0(z)] = \frac{d_1}{(1-\delta_1 z)^2} A_1(z)$$

and

$$R_0[f_n(z)] = \prod_{k=1}^{n-1} \left(\frac{d_k}{(1-\delta_k z + W_{k+1}(z))^2} \right) \frac{d_n z^n}{1-\delta_n z} A_{n+1}(z). \tag{3.5}$$

Since $\Lambda_0[W_k(z)] = O(z)$ we have $\Lambda_0 \left[\frac{d_k}{1-\delta_k z + W_{k+1}(z)} \right] = O(1)$ and hence

$$\Lambda_0[R_0[f_n(z)]] = O(z^n). \tag{3.6}$$

Therefore, by Definition 1, the δ- fraction solution given by the algorithm in Theorem 5 is a formal continued fraction solution to an admissible Riccati equation of the form (2.1). ∎

The purpose of the next theorem is to establish the connection between the formal power series solution and a formal δ- fraction solution.

Theorem 7. *If $R_0[W_0(z)] = 0$ is an admissible Riccati equation, then a formal δ- fraction solution corresponds to the unique fps solution that vanishes at $z = 0$.*

Proof. We prove the result for Riccati equations of the form (2.1). The proof is completely analogous in the other case. It is easy to establish the assertion that an admissible Riccati equation of the form (2.1) has a unique formal power series solution that vanishes at $z = 0$. From Theorem 2 we see that the relationship between a δ- fraction and its corresponding power series , $L(z)$, can be characterized by the equation

$$L(z) - \Lambda_0\left(f_n(z)\right) = O(z^{n+1}). \tag{3.7}$$

Thus,

$$\left\{ \begin{aligned} R_0[L(z)] &= R_0[(L(z) - f_n(z)) + f_n(z)] \\ &= R_0[f_n(z)] + (L(z) - f_n(z))B_0(z) + (L^2(z) - f_n^2(z))C_0(z) \\ &\quad -(L'(z) - f_n'(z)) \\ &= O(z^n) + O(z^{n+1}) + O(z^{n+1}) - O(z^n) \\ &= O(z^n) \end{aligned} \right. \tag{3.8}$$

(where the derivative of $L(z)$ is taken formally). This holds for every $n \in \mathbf{Z}^+$ and hence $L(z)$ must be the unique fps solution that vanishes at $z = 0$. ∎

Corollary 8. *Let $R[W(z)] = 0$ be an admissible Riccati equation. The δ- fraction solution guaranteed by Theorem 5 is the unique formal δ- fraction solution to the initial value problem*

$$\begin{cases} R[W(z)] = 0 \\ W(0) = 0. \end{cases} \tag{3.9}$$

Proof. This is an immediate consequence of the uniqueness of the fps solution and Theorem 2. ∎

A consequence of this corollary is given in the following corollary.

Corollary 9. *An admissible Riccati equation has a rational solution that vanishes at $z = 0$ if and only if its formal δ- fraction solution is a terminating continued fraction.*

Proof. This result is easily obtained from Corollary 8 and Theorem 3. ∎

Our next theorem provides the connection between the formal δ- fraction solution and the unique analytic solution (if an analytic solution exists) that vanishes at $z = 0$.

Theorem 10. *Let $R[W(z)] = 0$ be an admissible Riccati equation. If the formal δ- fraction solution converges uniformly in a neighborhood of $z = 0$ to a function W(z), then W(z) is the unique solution of $R[W(z)] = 0$ that is analytic at $z = 0$ satisfying $W(0) = 0$.*

Proof. Let f_n be the n^{th} approximant of the δ- fraction solution. By Theorem 5.13 in [7], $W(z) = \lim_{n\to\infty} f_n(z)$ is analytic in a neighborhood of $z = 0$, and the Taylor series expansion of $W(z)$ is the power series $L(z)$ to which the δ- fraction corresponds at $z = 0$. By Theorem 7, $L(z)$ is a fps solution of $R[W(z)] = 0$ at $z = 0$. It is therefore a solution of the Riccati equation in the neighborhood of $z = 0$ in which it converges. The assertion follows from the fact that $W(z) = L(z)$ for z in this neighborhood. ∎

4. Comments on Computational Aspects

One motivation for considering δ- fraction solutions is the desire for a more computationally satisfying algorithm than the C- fraction algorithm in [3,14]. A C- fraction is a finite or infinite continued fraction of the form

$$\frac{d_1 z^{e_1}}{1} + \frac{d_2 z^{e_2}}{1} + \cdots + \frac{d_n z^{e_n}}{1} + \cdots \tag{4.1}$$

where $d_n \in \mathbf{C}\backslash\{0\}$ and $e_n \in \mathbf{Z}^+$. A C- fraction is said to be a *regular* C- fraction if $e_n = 1$ for $n \in \mathbf{Z}^+$. The following two theorems from [3] are included to facilitate the comparison between δ- fraction solutions and C- fraction solutions.

Theorem 11. *(A) If $R[W(z)] = 0$ is an admissible Riccati equation of the form*

$$R[W(z)] := A(z) + B(z)W(z) + C(z)W^2(z) - W'(z) = 0 \tag{4.2}$$

that has a regular C- fraction solution, then d_n is a function of the first n coefficients of $A(z)$ and $B(z)$ and the first $n-1$ coefficients of $C(z)$.

(B) If $R[W(z)] = 0$ is an admissible Riccati equation of the form

$$R[W(z)] := zA(z) + B(z)W(z) + C(z)W^2(z) - z^k W'(z) = 0 \tag{4.3}$$

that has a regular C- fraction solution, then d_n is a function of the first n coefficients of $A(z)$, the first $n-1$ coefficients of $B(z)$ and the first $n-2$ coefficients of $C(z)$.

One significant consequence of this is that when computing, one may approximate the coefficient functions by polynomials without introducing any error into the computations due to these approximations. In other words, in order to calculate accurately the first n elements of the continued fraction, all one needs to use are the $(n-1)^{st}$ degree Taylor polynomials for $A(z)$ and $B(z)$ and the $(n-2)^{nd}$ degree Taylor polynomial (at $z=0$) for $C(z)$.

Another consequence is that one can considerably reduce the computing by calculating fewer terms of the coefficient functions at each step. In order to calculate $d_1, \ldots, d_n$, we start with the $(n-1)^{st}$ degree Taylor polynomials for $A(z)$ and $B(z)$ and the $(n-2)^{nd}$ degree Taylor polynomials for $C(z)$. At each step, the degree of the polynomials is reduced by one unit until at the last step just the constant terms of $A(z)$ and $B(z)$ are calculated. These two terms are all that are needed to calculate d_n. Thus, the program is quite efficient.

When the C- fraction solution to the Riccati equation is not a regular C- fraction the computational story is more complex as is illustrated by the next theorem.

Theorem 12. *(A) If $R[W(z)] = 0$ is an admissible Riccati equation of the form (4.2) then the equation will have a C- fraction solution. If we define*

$$\beta_n := \sum_{j=1}^{n} e_j \quad \text{and} \quad \gamma_n := \sum_{j=2}^{n} e_j \tag{4.4}$$

then d_n is a function of the first β_n coefficients of $A(z)$, the first $\gamma_n + 1$ coefficients of $B(z)$ and the first $\gamma_n + 1 - \beta_1$ coefficients of $C(z)$.

(B) If $R[W(z)] = 0$ is an admissible Riccati equation of the form (4.3), then the equation has a C- fraction solution where d_n is a function of the first β_n coefficients of $A(z)$, the first γ_n coefficients of $B(z)$ and the first $\gamma_n - \delta_1$ coefficients of $C(z)$.

From this theorem, it is clear that if the C- fraction is not a regular C- fraction, then the numbers of coefficients needed to calculate d_n accurately are functions of the exponents $e_1, e_2, \dots, e_n$. Therefore, one cannot be sure of the degrees of the polynomials to use in approximating the coefficient functions $A(z)$, $B(z)$ and $C(z)$ without a priori knowledge of the exponents in the continued fraction.

The δ- fraction solutions introduced in this paper have computational characteristics that are analogous to those in the case that the Riccati equation has a regular C- fraction solution. Before we proceed it is worth noting that every function analytic at $z = 0$ has both a regular δ - fraction expansion and a C- fraction expansion. These are identical in the case of a *regular* C- fraction expansion. The δ - fraction would then have $\delta_n = 0$ for $n = 1, 2, \dots$ and the C- fraction would have $e_n = 1$ for $n = 1, 2, \dots$.

Theorem 13. *(A) If $R[W(z)] = 0$ is an admissible Riccati equation of the form (4.2), then in the δ- fraction solution, δ_0 is a function of the constant term in $A(z)$, and for $n \in \mathbf{Z}^+$ d_n is a function of the first n coefficients in $A(z)$, the first $n - 1$ coefficients in $B(z)$,the first $n - 2$ coefficients in $C(z)$, and δ_n is a function of the first $n + 1$ coefficients in $A(z)$, the first n coefficients in $B(z)$, and the first $n - 1$ coefficients in $C(z)$. (If the number of coefficients is negative, then we assume there is no dependance on the coefficients.)*

(B) If $R[W(z)] = 0$ is an admissible Riccati equation of the form (4.3), then in the δ- fraction solution, δ_0 is a function of the constant term in $A(z)$, and for $n \in \mathbf{Z}^+$ d_n is a function of the first n coefficients in $A(z)$, $B(z)$,and $C(z)$, and δ_n is a function of the first

$n+1$ coefficients in $A(z)$, $B(z)$, and $C(z)$. (If the number of coefficients is negative, then we assume there is no dependance on the coefficients.)

Proof. (A) Let

$$\begin{cases} A_n(z) = a_{0,n} + a_{1,n}z + \ldots + a_{k,n}z^k + \ldots, \\ B_n(z) = b_{0,n} + b_{1,n}z + \ldots + b_{k,n}z^k + \ldots, \quad \text{and} \\ C_n(z) = c_{0,n} + c_{1,n}z + \ldots + c_{k,n}z^k + \ldots . \end{cases} \tag{4.5}$$

An inductive argument gives us that

(i) $a_{k,n}$ is a function of $a_{0,0}, \ldots, a_{k+n-1,0}, b_{0,0}, \ldots, b_{k+n-2,0}$ and $c_{0,0}, \ldots, c_{k+n-3,0}$ for $k = 0,1,\ldots, n = 1,2,\ldots$. (If $k+n-2 < 0$ or $k+n-3 < 0$, then there is no dependence on the coefficients of $B(z)$ or $C(z)$, respectively.)

(ii) $b_{k,n}$ and $c_{k,n}$ are functions of $a_{0,0}, \ldots, a_{k+n-2,0}$, $b_{0,0}, \ldots, b_{k+n-3,0}$ and $c_{0,0}, \ldots, c_{k+n-4,0}$ for $k = 0,1,\ldots, n = 1,2,\ldots$. (If $k+n-2 < 0$, $k+n-3 < 0$ or $k+n-4 < 0$, then there is no dependence on the coefficients of $A(z)$,$B(z)$ or $C(z)$, respectively.) The result now follows from the definitions of $\delta_n, n = 0,1,\ldots$ and $d_n, n = 1,2,\ldots$ given in the algorithm appearing in the proof of Theorem 5.

(B) The proof is analogous. ∎

As a result of this theorem, it is easy to see that the δ- fraction solutions have the same computational bonuses that regular C- fraction solutions have. The coefficient functions can be approximated by polynomials of appropriate degrees without introducing error into the computations up to a given n and the computations can be made very efficient (both as in the discussion following Theorem 11.) Thus, in the case when a Riccati equation does not have a regular C- fraction solution we can still enjoy the computational advantages associated with them if we turn to δ- fraction solutions instead of C- fraction solutions. Another advantage of using δ- fractions is that many convergence results are available for δ- fractions [11], whereas with C- fractions, virtually all of the convergence results apply to the special class of C- fractions, the regular C- fractions.

In closing, one more observation should be made. To the best of my knowledge, to date all of the δ- fraction expansions of functions have been obtained from known expansions in other types of continued fractions. This algorithm gives a method for obtaining δ- fraction expansions independent of other known expansions and independent of power series expansions. Clearly, the usefulness in this regard is restricted to functions that are solutions to admissible Riccati equations.